A StockOpter® *Insight* White Paper
From: Net Worth Strategies. Inc.

Black-Scholes Value and Employee Stock Options

(Part II)

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This section builds upon the material presented in *Black-Scholes Value and Employee Stock Options Part I*. It gives the actual BSV equation and explores some of the associated issues in somewhat more depth.

Variables and Equations

Let:

- C = the value of a call stock option.
- S = the current price of the stock.
- X = the strike price, also called the exercise price.
- r = the continuously-compounded annualized risk-free interest rate.
- q = the continuously-compounded annualized dividend yield of the stock. q is used in the Merton extension of the Black-Scholes model.
- t = the time in years until the option expires.
- σ = the volatility of the stock price.
- N(z) = the cumulative standard normal distribution evaluated at z. N(z) is the probability that a normally distributed variable is less than z standard deviations from its mean, where z can be positive or negative.

The Black-Scholes model says that

$$C = S\mathrm{N}(d_1) - Xe^{-rt}\mathrm{N}(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

and

$$d_2 = \frac{\ln\left(\frac{S}{X}\right) + \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$$
.

Extensions for Stocks that Pay Dividends

There are two simple extensions of the Black-Scholes model that take dividend payments into account. One of these substitutes *S* minus the present value, discounted at the risk-free rate, of the dividends to be paid during the remaining life of the option for *S*, both in the overall equation and in the equations for d_1 and d_2.

This can be fine-tuned a bit further by incorporating tax effects. The price of a stock typically goes down by somewhat less than the dividend amount when the dividend is paid due to tax effects. The amount by which *S* is reduced in the equations should be the present value of the expected decreases in price at the dividend payment dates.

The other simple extension of the Black-Scholes model to account for dividends was developed by Robert Merton for stocks that have a continuous dividend yield. This is a reasonable approximation for stock indices, such as the S&P 500. The Black-Scholes model as modified by Merton says that

$$C = Se^{-qt}\mathrm{N}(d_1) - Xe^{-rt}\mathrm{N}(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r - q + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

and

$$d_2 = \frac{\ln\left(\frac{S}{X}\right) + \left(r - q - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$$
.

Note that this model reduces to the original model when $q = 0$, i.e., the stock does not pay dividends, as it should.

StockOpter *Insight* uses the Merton extension of the Black-Scholes model. It does so for two reasons:

- The user does not have to input a dividend schedule.
- The remaining life of an ESO is often quite long.

This remaining life of the option being long makes the assumption that the stock will continue to pay the same dividend yield that it pays today, i.e., that the dividend will continue to be the same percentage of the stock price at the time of payment that it is today, somewhat less problematic than the assumption that it will continue to pay the same dividend amount that it pays today. It also makes the assumption that dividends are paid continuously less of a problem, as discussed below.

The Merton extension is still valid if dividends are not paid continuously, so long as q is the average annualized dividend yield for the remaining life of the option. The annualized dividend yield today, which StockOpter *Insight* uses for q, is a reasonable estimate of q when the remaining life is relatively long. When the remaining life is relatively short, it may or may not be a good estimate, depending on where the date for which one is calculating the value falls in the cycle of dividend payment dates. The estimate will be best just after a dividend payment, and worst just before a payment date. The bias is in the direction of understating q, and thus overstating the BSV and the TV.

However, it should be born in mind that the TV will tend to be largest when the remaining life of the option is relatively long, in which case the bias due to using the current annual dividend yield as our estimate of q will be relatively small. When the remaining life of the option is relatively short, the TV will generally be relatively small, and thus the dollar errors in TV will generally be relatively small even though the percentage errors may be relatively large.

Additional Assumptions

Some of the assumptions underlying the Black-Scholes model were discussed in the INTRODUCTORY LEVEL material. Some additional assumptions are that:

- The continuously-compounded risk-free interest rate r is constant, the same for all maturities, and known.
- The volatility σ is constant and known.
- Security trading is continuous.

Each of these assumptions is discussed in more detail below. In addition, the discussion of the lognormal random walk model already given in the INTRODUCTORY LEVEL material is expanded upon slightly below. As noted in the INTRODUCTORY LEVEL material, for a more detailed discussion of the lognormal random walk model please visit http://www.networthstrategies.com/. There are still more assumptions underlying the Black-Scholes model that are not worth going into in this document.

It turns out that the assumptions about the continuously-compounded risk-free rate r can be relaxed. The Black-Scholes model is correct even if interest rates can change and the changes are not known in advance, so long as the value used for r is the annualized continuously-compounded interest rate today for a zero-coupon government bond maturing on the expiration date of the option and we make another reasonable assumption not worth going into in this document.

In practice, ideally users of StockOpter *Insight* should use the yield of a zero-coupon treasury bond maturing somewhere in the middle of the maturity dates of the options, since only one interest rate is used. It is probably best to err in the direction of the more distant maturities, since the interest rate used will have a larger effect on options that mature farther into the future. It should also be born in mind that the approximations involved and the purposes to be served make it not worth the trouble of trying to be very precise. For the same reason, it is probably not worth converting interest rates to a continuously compounded basis. This is especially true when rates are low, as then the effect of the conversion on the rate is quite small.

It turns out that the assumption that the volatility σ is constant can be relaxed, though it still must be assumed to be known, now as a known function of time. The volatility is a kind of standard deviation, and the variance is the square of the standard deviation. If the volatility changes over the life of the option, in theory one should calculate the average variance over the life of the option and then use the corresponding volatility, i.e., the square root of the average variance, in calculating BSV. In practice, first, the result will be almost the same if one simply calculates the average volatility, and second, such a calculation will not normally be called for.

Rather, the usefulness of this result is that if one sees evidence that volatility is not expected to be constant, *this is not a problem*. An example of such evidence is seeing an implied volatility of 30% for a 6-month option and 25% for a one-year option. This implies that the volatility is expected to be about 20% for the time interval from 6 months from now to one year from now, as 30% and 20% average to 25%. Since we should use the average anyway, we would use 30% for a 6-month option, 25% for a 1-year option, or, for that matter, 27.5% for a 9-month option. However, even that amount of precision is likely to be overkill, particularly since StockOpter

Insight has only one volatility input, which must be used for options of varying remaining times to maturity.

The assumption that trading is continuous can also be relaxed. It turns out that most price change occurs during trading hours, so in principle we can account for this by keeping track of trading time rather than calendar time. In practice, however, the difference in the remaining life in years of an option calculated as remaining trading days divided by trading days in a year versus calculated as remaining calendar days divided by calendar days in a year will be inconsequential for the purposes of StockOpter *Insight*.

The lognormal random walk model of stock price changes assumes that prices change continuously, without any virtually instant jumps up or down. In practice, we know that such jumps occur every once in a while. However, the effects of such jumps damp out as one lengthens the time interval considered. It is reasonable to disregard such effects unless one is trying to be much more precise than is necessary for the purposes of StockOpter *Insight*.

Validity of the Black-Scholes Model

The Black-Scholes model is not perfect. Some of its assumptions are not entirely met in practice, some instances of which are discussed above. Also, if one solves the actual prices of stock options for the volatility, which is how implied volatilities are calculated, the volatility for a given time to maturity depends somewhat on the strike price. There are reasonable explanations for this phenomenon, but it still reveals imperfections in the model as a description of real-world pricing.

Nevertheless, the Black-Scholes model is simpler than other models, yet more than accurate enough for our purposes as a pricing model for non-ESO stock options. Any inaccuracies due to other reasons pale next to the uncertainty of how to adjust the value for the fact that an option is an ESO. Still, as noted at the end of the Part I section of this paper, time values calculated from Black-Scholes values can be a useful input to decisions as to whether to exercise ESOs.